Global operator expansions in conformally invariant relativistic quantum field theory

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(Received 15 October 1974)

We establish global conformal operator expansions in the Minkowski region in several models and discuss their formulation in the general theory. Whereas the vacuum expansions are termwise manifestly conformal-invariant, the expansions away from the vacuum do not share this property.

I. INTRODUCTION

Conformally covariant operator expansions introduced some time ago[1-3] have been recently reinvestigated by Mack[4] in the Euclidean region from the point of view of harmonic analysis over the group SO($D+1,1$), D being the space-time dimension.

Representing relations between Euclidean (Schwinger) functions one may write

$$A_{[\alpha]}(x)B_{[\alpha]}(y)=\sum_N \int K^{[N]}_{[\alpha,\beta,\gamma]}(x-z,y-z)\times C^{[N]}_{[\gamma]}(z)\,d^4z\,, \quad (1.1)$$

where A, B, and C are conformally covariant type Ia (Ref. 5) Euclidean fields and $[\alpha]$, $[\beta]$, $[\gamma]$ are sets of indices characterizing their tensor nature. As identities between ($n+1$)- and n-point Schwinger functions, (1.1) can be unambiguously continued to the Minkowski region as corresponding identities between time-ordered functions. From those, relations between the Wightman functions of the theory follow[6] which contain in principle the Minkowski analog of the expansion (1.1). However, the relation between the Minkowski expansion and Minkowski conformal invariance has never been very clear because of an incomplete understanding of the action of finite conformal transformations on quantized fields. The latter problem has been investigated in Ref. 7 for free fields and more recently[8,9] in the general case.

It turns out that in quantum field theory the transition from SO($D+1,1$) to SO($D,2$), in going from the Euclidean to the Minkowski region, leads one to representations of the universal covering group of the conformal group, and the transformation properties of a local field $A(x)$ are most conveniently characterized in terms of nonlocal components $A^{\xi}(x)$, with

$$A(x)=\int_0^1 d\xi A^{\xi}(x)\,. \quad (1.2)$$

Equation (1.2) represents a decomposition with respect to a central element Z of the universal covering of the conformal group such that

$$ZA^{\xi}(x)Z^{-1}=\exp[-i\pi(d_a-2\xi)]A^{\xi}(x)\,, \quad (1.3)$$

d_a being the dimension of the (scalar) field $A(x)$, and for special conformal transformations

$$x\rightarrow x_T=\frac{x-bx^2}{\sigma(b,x)} \quad (1.4)$$

we have

$$U(b)A^{\xi}(x)U^{-1}(b)=\frac{1}{[\sigma_+(b,x)]^{d_a-\xi}[\sigma_-(b,x)]^{\xi}}A^{\xi}(x_T)\,, \quad (1.5)$$

with

$$\sigma(b,x)=1-2b\cdot x+b^2x^2$$

and

$$[\sigma_\pm(b,x)]^\lambda=(-b^2\mp i\epsilon b_0)^\lambda\left[-\left(x-\frac{b}{b^2}\right)^2\mp i\epsilon\left(x_0-\frac{b_0}{b^2}\right)\right]^\lambda \quad (1.6)$$

being the analytic continuation of the corresponding Euclidean expression from the respective positive and negative imaginary values of the b_0, x_0 variables. For the remaining transformations of the conformal group the $A^{\xi}(x)$ transform conventionally.

In the present paper we will reexamine, in the light of our previous results,[8] the problem of conformal operator expansions directly in the Minkowski region.

Because of their simpler transformation properties the $A^{\xi}(x)$ stand out as natural objects in terms of which such an expansion should be formulated. We therefore expect instead of (1.1) the following expansion:

$$A^{\xi_a}_{[\alpha]}(x)B^{\xi_b}_{[\beta]}(y)=\sum_N\int K^{[N],\xi_a,\xi_b}_{[\alpha,\beta,\gamma]}(x-z,y-z)\times C^{[N],\xi_c}_{[\gamma]}(z)\,d^4z\,. \quad (1.7)$$

The $A^{\xi}(x)$ generalize the notion of the creation and annihilation parts of a free field to the interacting case: As a consequence of the spectrum condition and the transformation law (2.4) we find

$$A(x)|0\rangle=A^0(x)|0\rangle\,,$$
$$A^{\dagger}(x)|0\rangle=(A^{d_a \bmod(1)}(x))^{\dagger}|0\rangle\,. \quad (1.8)$$

Whenever the expansion (1.7) is applied to the vacuum state (vacuum expansion) only the components $\xi_b=\xi_c=0$ participate as a result of (1.8).

We will prove the validity of the vacuum expansions for free fields in Sec. II. In Sec. III we investigate the vacuum expansion in the general case, and in Sec. IV we test our results in a soluble model. We devote Sec. V to a discussion of the more general expansion with ξ_b and ξ_c different from zero.

II. THE VACUUM EXPANSION FOR FREE FIELDS

In this section we show the existence of a vacuum expansion for free fields. Let $A(x)$, $B(x)$ be massless free fields. For definiteness we will consider explicitly only the case of scalar fields in 4 space-time dimensions, the generalization to arbitrary values of spin and dimension being clear. For free fields the ξ decomposition (1.2) leads to only two components, the creation parts $A^0(x)$, $B^0(x)$, and the annihilation parts $A^1(x)$, $B^1(x)$. Consider the two-particle state

$$|P,q\rangle=a^{\dagger}(p_1)b^{\dagger}(p_2)|0\rangle\,, \quad (2.1)$$

with

$$P=p_1+p_2,\quad q=p_1-p_2$$

and

$$A^0(x)=\frac{1}{(2\pi)^{3/2}}\int a^{\dagger}(p)\delta_+(p^2)e^{ipx}\,d^4p\,,$$

$$B^0(x)=\frac{1}{(2\pi)^{3/2}}\int b^{\dagger}(p)\delta_+(p^2)e^{ipx}\,d^4p\,.$$

The traceless symmetric conformal tensors bilinear in A, B have canonical dimensions and therefore[10] are conserved. With a suitable normalization their matrix elements are

$$\langle P,q|C^{\mu_1\cdots\mu_n}(0)|0\rangle=q^{\mu_1}\cdots q^{\mu_n}-\mathrm{Tr}'\,, \quad (2.2)$$

where the Tr′ operation consists in removing the traces with the help of a metric tensor

$$\tilde{\delta}^{\mu\nu}=\delta^{\mu\nu}-\frac{P^{\mu}P^{\nu}}{P^2}\,. \quad (2.3)$$

For instance,

$$q^{\mu_1}q^{\mu_2}-\mathrm{Tr}'=q^{\mu_1}q^{\mu_2}-\tfrac{1}{3}\tilde{\delta}^{\mu_1\mu_2}q^2\,.$$

Since

$$P^2=-q^2,\quad P\cdot q=0\,, \quad (2.4)$$

it is clear that (2.2) corresponds to matrix elements of local conserved and traceless operators.

Consider now the state (2.1) in the center-of-mass frame $P=(P_0,0,0,0)$. We want to write

$$\langle P_0'\vec{q}\,'|P_0\vec{q}\rangle=\sum_n a^{\mu_1\cdots\mu_n}(P_0,\vec{q})\times\langle P_0'\vec{q}\,'|\tilde{C}_{\mu_1\cdots\mu_n}(P_0)|0\rangle. \quad (2.5)$$

The matrix elements of the C's in this case, up to a normalization factor, are the spherical harmonics in $\vec{q}$. Using the completeness of spherical harmonics one gets

$$a^{i_1\cdots i_n}(P_0,\vec{q})=P_0^{-2n}\left[\frac{(2n-1)!!}{n!}\right]^2\frac{2n+1}{32\pi^5}\times(q^{i_1}\cdots q^{i_n}-\mathrm{Tr}')\,, \quad (2.6)$$

the time components of the coefficients being zero, so that

$$|P_0,\vec{q}\rangle=\sum_n a^{i_1\cdots i_n}(P_0,\vec{q})\tilde{C}^{i_1\cdots i_n}(P_0)|0\rangle \quad (2.7)$$

and by Lorentz invariance

$$|P,q\rangle=\sum_n a^{\mu_1\cdots\mu_n}(P,q)\tilde{C}_{\mu_1\cdots\mu_n}(P)|0\rangle\,, \quad (2.8)$$

with

$$a^{\mu_1\cdots\mu_n}(P,q)=(P^2)^{-n}\left[\frac{(2n-1)!!}{n!}\right]^2\frac{2n+1}{32\pi^5}\times(q^{\mu_1}\cdots q^{\mu_n}-\mathrm{Tr}')\,. \quad (2.9)$$

From (2.8) we finally get

$$A^0(x)B^0(y)|0\rangle=\sum_N\int K^{00}_{[N]}(x-z,y-z)\,C^0_{[N]}(z)\,d^4z\,|0\rangle\,, \quad (2.10)$$

with

$$K^{00}_{[N]}(x-z,y-z)=\pi\int d^4P\,d^4q\,e^{iP[(x+y)/2-z]}e^{iq(x-y)/2}\,\delta(P^2+q^2)\delta(P\cdot q)\theta(P^2)\theta(P^0)a^{\mu_1\cdots\mu_n}(P,q)\,. \tag{2.11}$$

Equation (2.10) proves the vacuum expansion for free fields. If we take equal fields, statistics will restrict the summation to even tensor fields. Furthermore, in this case in addition to (2.10) we will have

$$A^1(x)A^0(y)|0\rangle=\int\Delta^+(x-y)\delta^4(\tfrac{1}{2}(x+y)-z)\underline{1}\,d^4z\,|0\rangle. \tag{2.12}$$

III. THE GENERAL VACUUM EXPANSION

In the previous section we described the vacuum expansion for free fields. It is clear that, in general, the validity of the vacuum expansion is equivalent to the existence of local fields $C_{[N]}(x)$ creating from the vacuum the subspace of states carrying any irreducible unitary (ray) representation of the conformal group in the Hilbert space. In this case we should have

$$A^{\xi_a}(x_1)B^0(x_2)|0\rangle=\sum_N\int K^{\xi_a,0}_{[N]}(x_1-x_3,x_2-x_3)\times C^0_{[N]}(x_3)\,d^4x_3|0\rangle\,. \tag{3.1}$$

The kernels in (3.1) can be determined by considering the transformation law of both sides under the conformal group. The orthogonality between states belonging to different unitary (ray) representations allows one to enforce the correct properties term by term in the left-hand side of (3.1).

Concentrating on scalar fields we have

$$A^{\xi_a}(x_1)B^0(x_2)|0\rangle=\int\frac{N_{abc}C^0(x_3)|0\rangle\,d^4x_3}{(-x_{12}{}^2)^{(\lambda_1,\xi_1)}(-x_{13}{}^2)^{(\lambda_2,\xi_2)}(-x_{23}{}^2)^{(\lambda_3,\xi_3)}}+\cdots, \tag{3.2}$$

with

$$\frac{1}{(-x^2)^{(\lambda,\xi)}}=\frac{1}{(-x^2{}_+)^{\lambda-\xi}(-x^2{}_-)^{\xi}} \tag{3.3}$$

and

$$x_{ij}=x_i-x_j\,,$$
$$x^2{}_\pm=(x_0\pm i\epsilon)^2-\mathbf{x}^2\,,$$

N_{abc} = normalization constant.

Using the transformation law (1.5) and

$$[-(x-y)^2{}_\pm]^\lambda=[\sigma_\pm(b,x)]^\lambda[-(x_T-y_T)^2{}_\pm]^\lambda\times[\sigma_\mp(b,y)]^\lambda\,, \tag{3.4}$$

with x_T given by (1.4) and $(\sigma_\pm)^\lambda$ given by (1.6), one obtains, by comparing both sides of (3.2) after the conformal transformation,

$$\lambda_1=\tfrac{1}{2}(d_a+d_b+d_c)-2\,,\quad \lambda_2=\tfrac{1}{2}(d_a-d_b-d_c)+2\,,\quad \lambda_3=\tfrac{1}{2}(d_b-d_a-d_c)+2\,, \tag{3.5}$$

and

$$\xi_a=(\xi_1+\xi_2)\,\mathrm{mod}(1)\,,\quad 0=(\xi_3-\xi_1+\lambda_1)\,\mathrm{mod}(1) \tag{3.6}$$

with the dimension d_c being related to ξ_a by

$$\xi_a-\tfrac{1}{2}(d_a+d_b-d_c)=0\,\mathrm{mod}(1)\,. \tag{3.7}$$

Equation (3.6) still leaves one with an undetermined parameter, say ξ_1. This freedom is removed by exploiting the spectrum condition: The state $A^{\xi_a}(x_1)B^0(x_2)|0\rangle$ can be analytically continued to positive imaginary values of the time variable x_2^0. This is only compatible with the representation (3.2) if

$$\xi_3=0\,,\quad \xi_1=\lambda_1\,\mathrm{mod}(1)\,,\quad \xi_2=(\xi_a-\lambda_1)\,\mathrm{mod}(1)\,. \tag{3.8}$$

It should be noticed that the kernel in (3.2) is a perfectly well-defined distribution as long as λ_2 is not a positive integer. This is quite clear since (3.3) can be written as

$$\frac{1}{(-x^2)^{(\lambda,\xi)}}=\frac{1}{(-x^2{}_-)^\lambda}+2\frac{i\sin\pi(\lambda-\xi)}{(x^2)^\lambda}(x^2)\times[e^{-i\pi\xi}\theta(x_0)-e^{i\pi\xi}\theta(-x_0)]\,, \tag{3.9a}$$

which is a tempered distribution defined by analytic continuation from $\lambda<1$ by means of

$$\int\frac{e^{iP\cdot x}d^4x}{(-x^2)^{(\lambda,\xi)}}=-|P^2|^{\lambda-2}2^{5-2\lambda}\pi\Gamma(2-\lambda)\Gamma(1-\lambda)\times[\theta(P^2)\theta(P_0)\sin^2\pi\xi+\theta(P^2)\theta(-P_0)\sin^2\pi(\lambda-\xi)-\theta(-P^2)\sin\pi\xi\,\sin\pi(\lambda-\xi)]\,. \tag{3.9b}$$

For positive integer λ_2 we have a degenerate case with the kernel in (3.2) ill defined.

In order to check the consistency of the vacuum expansion (3.1), one has to show that it reproduces the conformal-invariant three-point function in the whole Minkowski space:

$$\langle 0|C^*(x_3)A^{\xi a}(x_1)B(x_2)|0\rangle$$
$$=\langle 0|C^*(x_3)A(x_1)B(x_2)|0\rangle$$
$$=g_{abc}[-(x_{21})^2{}_+]^{-\delta_3}[-(x_{23})^2{}_+]^{-\delta_1}[-(x_{13})^2{}_+]^{-\delta_2}, \tag{3.10}$$

with

$$\delta_1=\tfrac{1}{2}(d_b+d_c-d_a), \quad \delta_2=\tfrac{1}{2}(d_a+d_c-d_b),$$
$$\delta_3=\tfrac{1}{2}(d_a-d_b-d_c).$$

In case the composite fields create a complete set of states from the vacuum, this would even be a proof of the expansion. As a by-product of this calculation one obtains also the normalization constant N_{abc}.

Inserting the expansion (3.1) into the left-hand side of (3.10) and using the field normalization

$$\langle 0|C^*(x_1)C(x_2)|0\rangle=[-(x_{21})^2{}_+]^{-d_c} \tag{3.11}$$

we get

$$\langle 0|C^*(x_2)A^{\xi a}(0)B(x_1)|0\rangle$$
$$=\frac{N_{abc}}{[-(x_{12})^2{}_+]^{\lambda_1}}\int d^4x_3\,\frac{[-(x_3)^2{}_-]^{\lambda_3-2}[-(x_3)^2{}_+]^{d_c-2}}{[-(x_{31})^2{}_-]^{\lambda_3}[-(x_{32})^2{}_+]^{d_c}}. \tag{3.12}$$

It suffices to calculate the integral in (3.12) for

$$\vec{x}_2=0, \quad \mathrm{Im}\,x_2^0<0, \quad \mathrm{Im}\,x_1^0>0$$

and then use Lorentz invariance and the analyticity properties of the integral to obtain the general result. Performing the angular integration and introducing new variables (for details see the Appendix)

$$u=x_3^0+|x_3|, \quad v=x_3^0-|x_3|,$$
$$\xi=x_1^0+|x_1|, \quad \eta=x_1^0-|x_1|, \quad z=x_2^0,$$

one can express the integral in (3.12) in terms of one-dimensional integrals I^n as

$$\langle 0|C^*(x_2)A^{\xi a}(0)B(x_1)|0\rangle=\frac{\pi N_{abc}}{[-(x_1)^2{}_+]^{\lambda_1}2(\lambda_3-1)(\xi-\eta)}[I^1(\eta,z)I^0(\xi,z)-I^0(\eta,z)I^1(\xi,z)], \tag{3.13}$$

where

$$I^n(\xi,z)=\int_{-\infty}^{+\infty}du\,u^n\frac{[-(u-i\epsilon)]^{\lambda_3-2}[-(u+i\epsilon)]^{d_c-2}}{[-(u-\xi-i\epsilon)]^{\lambda_3-1}[-(u-z+i\epsilon)]^{d_c}}. \tag{3.14}$$

For $n=0,1$ one can evaluate (3.14) as a contour integral in terms of hypergeometric functions[11]

$$I^n(\xi,z)=2i\pi(\lambda_3-1)^n(-1)^n\frac{\Gamma(\lambda_3+d_c+n-3)}{\Gamma(d_c+n-1)\Gamma(\lambda_3)}e^{-i\pi\lambda_3}\frac{(-\xi)^{d_c+n-2}(-z)^{\lambda_3-2}}{[-(\xi-z)]^{\lambda_3+d_c-2}}F(n-1,2-\lambda_3;d_c+n-1;\xi/z), \tag{3.15}$$

so that finally comparing (3.13) with (3.10) one gets

$$N_{abc}2\pi^2\frac{\Gamma(\lambda_3+d_c-2)\Gamma(\lambda_3+d_c-3)}{\Gamma(d_c)\Gamma(d_c-1)\Gamma(\lambda_3)\Gamma(\lambda_3-1)}=g_{abc}. \tag{3.16}$$

Considering now an arbitrary 3-point function and its transformation under Z,[8] we obtain

$$\langle 0|D(x_1)B^0(x_2)F^0(x_3)|0\rangle=0 \tag{3.17}$$

unless

$$\tfrac{1}{2}(t_f+t_b-t_d)=0\,\mathrm{mod}(1), \tag{3.18}$$

with $t=d-s$. In an interacting theory we may expect that (3.18) does not have any solutions since the dimensions of composite fields should not be additive. This being the case, we conclude from (3.17) and (3.1)

$$B^0(x_2)F^0(x_3)|0\rangle=0. \tag{3.19}$$

Equation (3.19) allows one to elevate (3.1) to a true operator expansion,

$$A^{\xi a}(x_1)B^0(x_2)=\sum_N\int K^{\xi a,0}_{[N]}(x_1-x_3,x_2-x_3)\times C^0_{[N]}(x_3)\,d^4x_3. \tag{3.20}$$

Let us finally remark that for free fields, although λ_2 in (3.2) is 1 and ξ_2 from (3.8) is 0, even in this degenerate case one can write the operator expansion in the form

$$A^0(x_1)B^0(x_2)=\frac{1}{32\pi^5}\int\frac{C^0(x_3)\,d^4x_3}{[-(x_{13})^2{}_+][-(x_{23})^2{}_+]}+\cdots, \tag{3.21}$$

with

$$C^0(x)=A^0(x)B^0(x) \tag{3.22}$$

as can be seen directly from (2.9) and (2.10).

IV. MODEL CALCULATIONS

We shall discuss in this section the vacuum expansion in the Thirring model. Since the Wightman functions of the model factorize in terms of the variables $u=x^0+x^1$, $v=x^0-x^1$, it is convenient to classify a general field $\mathcal{O}$ in terms of its u and v "dimension" instead of its dimensions and spin. In this way we have for a special conformal transformation

$$U(b)\mathcal{O}^{d_1d_2\xi_1\xi_2}(u,v)U^{-1}(b)=\frac{\mathcal{O}^{d_1d_2\xi_1\xi_2}(u/(1-\gamma u),v/(1-\delta v))}{(1-\gamma u_+)^{2(d_1-\xi_1)}(1-\gamma u_-)^{2\xi_1}(1-\delta v_+)^{2(d_2-\xi_2)}(1-\delta v_-)^{2\xi_2}}. \tag{4.1}$$

The factorized form (4.1) which also holds for Lorentz transformations and dilatations clearly corresponds to the fact that $O(2,2)=O(2,1)\times O(2,1)$.[12]

For simplicity we will consider a pure "u Thirring field." The factorization property allows us to reduce the general Thirring model to this case, which corresponds in Klaiber's[13] notation to

$$\psi(u)=\psi_1(u),\quad g=0,\quad a+b=2\lambda. \tag{4.2}$$

The Wightman functions of this model are just the free $d_1=\frac{1}{2}$ functions raised to a power $2d$:

$$\langle\psi_1(u_1)\cdots\psi_1(u_n)\psi_1^*(u_1')\cdots\psi_1^*(u_n')\rangle=\left[\frac{\prod_{i<j}(u_i-u_j)\prod_{i<j}(u_i'-u_j')}{\prod(u_i-u_j')}\right]^{2d} \tag{4.3}$$

so that we may equivalently describe such a Klaiber field by an "exponential model"

$$\psi(u)=\exp[i\sqrt{2d}\,\varphi^+(u)]\exp[i\sqrt{2d}\,\varphi^-(u)],$$

where $\phi(u)$ is the u part of a zero-mass two-dimensional free field, in complete analogy to the model discussed in Ref. 8. Consider now the vacuum expansion

$$\psi^*(u_1)\psi^*(u_2)|0\rangle=\sum_\xi\psi^{*\xi}(u_1)\psi^{*0}(u_2)|0\rangle=\sum_n\int K^{(n)}(u_1-u,u_2-u)\times\mathcal{O}^{(n)}(u)|0\rangle\,du, \tag{4.4}$$

where the arguments of the previous section fix the kernels to be

$$K^{(n)}=\frac{C_n}{(u_{12}-i\epsilon)^{\lambda_1}(u_1-u-i\epsilon)^{1-2\delta_n}(u_1-u+i\epsilon)^{\delta_n}(u_2-u+i\epsilon)^{1-\delta_n}}, \tag{4.5}$$

with

$$\lambda_1=2d+\delta_n-1,\quad \delta_n=\text{dimension of }\mathcal{O}^{(n)}.$$

Inserting (4.4) into the 4-point functions we obtain

$$\langle\psi(u_3)\psi(u_4)\psi^*(u_1)\psi^*(u_2)\rangle=\sum_n\frac{C_ng_nI_n}{(u_{34}-i\epsilon)^{2d-\delta_n}(u_{12}-i\epsilon)^{\lambda_1}} \tag{4.6}$$

with

$$I_n=\int_{-\infty}^{+\infty}\frac{du}{(u_3-u-i\epsilon)^{\delta_n}(u_4-u-i\epsilon)^{\delta_n}(u_1-u-i\epsilon)^{1-2\delta_n}(u_1-u+i\epsilon)^{\delta_n}(u_2-u+i\epsilon)^{1-\delta_n}} \tag{4.7}$$

and

$$\langle\psi(u_3)\psi(u_4)\mathcal{O}^{(n)}(u)\rangle=\frac{g_n}{(u_{34}-i\epsilon)^{2d-\delta_n}(u_3-u-i\epsilon)^{\delta_n}(u_4-u-i\epsilon)^{\delta_n}}. \tag{4.8}$$

Taking $u_2=0$ and the configuration $0<u_2<u_4<u_3$, an arbitrary configuration being obtained by analytic continuation from this one, we evaluate I_n by closing the integration contour in the upper u complex plane:

$$I_n=\exp[i\pi(1-2\delta_n)](-2i)\sin(\pi\delta_n)\,u_2^{-1+2\delta_n}u_3^{-\delta_n}u_4^{-\delta_n}\int_0^1\frac{du}{(u)^{1-\delta_n}(1-u)^{1-\delta_n}[1-(u_2/u_3)u]^{\delta_n}[1-(u_2/u_4)u]^{\delta_n}}. \tag{4.9}$$

The last integral is given by a generalized hypergeometric function

$$F(\delta_n, \delta_n, \delta_n, 2\delta_n; u_2/u_3, u_2/u_4),$$

which can be expressed in terms of an ordinary hypergeometric function.[10] After translating back by u_1 we obtain

$$I_n = -2i\sin(\pi\delta_n)\exp[i\pi(1-2\delta_n)] \times (u_{21})^{-1+2\delta_n}(u_{31})^{-\delta_n}(u_{42})^{-\delta_n} \times \frac{\Gamma^2(\delta_n)}{\Gamma(2\delta_n)} F(\delta_n, \delta_n, 2\delta_n; r), \tag{4.10}$$

with

$$r = \frac{(u_1-u_2)(u_3-u_4)}{(u_3-u_1)(u_4-u_2)}.$$

The decomposition (4.6) now takes the form

$$\langle\psi(u_3)\psi(u_4)\psi^*(u_1)\psi^*(u_2)\rangle = \sum_n \frac{\tilde{C}_n}{(u_{34})^{2d}(u_{12})^{2d}} r^{\delta_n} F(\delta_n, \delta_n, 2\delta_n; r), \tag{4.11}$$

where all the constants have been absorbed into $\tilde{C}_n$.

From conformal invariance the general form of the 4-point function is

$$\langle\psi(u_3)\psi(u_4)\psi^*(u_1)\psi^*(u_2)\rangle = \frac{1}{(u_{34})^{2d}(u_{12})^{2d}} f(r), \tag{4.12}$$

with $f(r)$ being given in the Thirring model [cf. (4.3)] by

$$f(r) = \left(\frac{r^2}{1-r}\right)^{2d}. \tag{4.13}$$

The composite operators $\mathcal{O}^{(n)}$ can be computed in this model by a limiting procedure in the manner of Lowenstein[14] or alternatively by writing the field $\psi(u)$ as in the exponential model,

$$\psi^*(u) = \exp[i\sqrt{2d}\,\phi^+(u)]\exp[i\sqrt{2d}\,\phi^-(u)] = e^*_{2d}(u), \tag{4.14}$$

so that

$$\mathcal{O}^0 = \lim_{\epsilon\to 0}\epsilon^{-2d}\tilde{e}_d(u)\tilde{e}_d(u-\epsilon) = \tilde{e}_{4d}(u), \quad \dim\mathcal{O}^0 = 4d$$

$$\hat{\mathcal{O}}^1 = :\tilde{e}_{4d}(u)\partial_u\phi(u): = \frac{i}{2\sqrt{d}}\partial_u\tilde{e}_{4d}(u), \quad \dim\hat{\mathcal{O}}^1 = 4d+1$$

$$\mathcal{O}^2 = :\tilde{e}_{4d}(u)\partial_u\phi(u)\partial_u\phi(u): - i2\sqrt{d}\,:\tilde{e}_{4d}(u)\partial_u^2\phi(u):, \quad \dim\mathcal{O}^2 = 4d+2$$

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and in general

$$\dim\mathcal{O}^{(2n)} = \delta_{2n} = 4d + 2n, \tag{4.15}$$

$$\dim\hat{\mathcal{O}}^{(2n+1)} = 4d + 2n + 1. \tag{4.16}$$

The operators $\hat{\mathcal{O}}^{(2n+1)}$ do not participate in the expansion (4.11) since they are derivatives of type Ia conformal "tensors" and therefore not type Ia themselves. Notice also that a linear trajectory for the dimensions of the irreducible conformal tensors, as in (4.15), is an atypical two-dimensional feature owing to the failure of the Parisi-Callan-Gross theorem[15] in this case.

Inserting (4.15) into the expansion (4.11) and comparing with (4.13) we have

$$\frac{1}{(1-r)^{2d}} = \sum_n r^n \tilde{C}_n F(4d+n, 4d+n; 8d+2n; r). \tag{4.17}$$

Since the hypergeometric function can be expanded in Taylor series for $|r|<1$ and so can the left-hand side of (4.17), the coefficients $\tilde{C}_n$ can be determined recursively by comparing both expansions, with $\tilde{C}_{2n+1}$ vanishing owing to the absence of $\hat{\mathcal{O}}^{(2n+1)}$ in the expansion (4.4). For arbitrary values of r the validity of (4.17) is assured by analytic continuation, thus proving the correctness of the vacuum expansion (4.4) at the 4-point-function level. The generalization of this result for an arbitrary n-point function and the related problem of the vacuum expansion for composite operators will be considered elsewhere.

The $\tilde{C}_n$ can also be determined as in Sec. III by using the expansion (4.4) in the 3-point function

$$\langle\mathcal{O}^{*(2n)}(u_3)\psi^*(u_1)\psi^*(u_2)\rangle = \int K^{(2n)}(u_1-u, u_2-u)\langle\mathcal{O}^{*(2n)}(u_3)\mathcal{O}^{(2n)}(u)\rangle\, du, \tag{4.18}$$

corresponding to the construction of the composite operators $\mathcal{O}^{*(2n)}$ from $\psi(u_3)\psi(u_4)$ in (4.11).

The consideration of the vacuum expansion for $\psi^*(u_1)\psi(u_2)|0\rangle$ leads to degenerate kernels since the dimensions of the relevant "tensor" operators are canonical in this case:

$$\dim\mathcal{O}^{(n)} = n, \quad \mathcal{O}^{(0)} = 1.$$

This case therefore should be treated by a limiting procedure taking $\dim\mathcal{O}^{(n)} = n+\epsilon$ and letting $\epsilon\to 0$ at the end of the calculation.

V. COMMENTS ON THE GENERAL EXPANSION

In this section we make some considerations on the more general expansion problem, i.e.,

$$A^{\xi_a}(x_1)B^{\xi_b}(x_2)=\sum_N \int K^{\xi_a\xi_b}_{[N]}(x_1-x,x_2-x)\times C^{\xi_c}_{[N]}(x)\,d^4x\,. \quad (5.1)$$

If (5.1) exists, the transformation law of both sides under Z, Eq. (1.3), fixes ξ_c to be

$$\xi_c=[\xi_a+\xi_b-\tfrac{1}{2}(t_a+t_b-t_c)]\bmod(1)\,, \quad (5.2)$$

with

$$t=d-s\,.$$

For the vacuum expansion one is able to obtain the kernels in the form (3.2) by imposing that the right-hand side has the correct conformal transformation law term by term. In the general case there is no good reason for such a requirement which would lead to incorrect results even for free fields. We cannot expect therefore to determine the general kernel *a priori* from a direct application of conformal invariance. Consider, for instance, the free field case. It suffices then to prove (5.1) to examine the 1-1 particle matrix element of the product of two fields, that is,

$$\langle p_2|A^0(x_2)A^d(x_1)|p_1\rangle=\frac{2}{(2\pi)^2}e^{-iPX}\,e^{-iqx} \quad (5.3)$$

with

$$P=p_1-p_2,\quad q=p_1+p_2\,,$$

$$X=\tfrac{1}{2}(x_1+x_2)\,,\quad x=\tfrac{1}{2}(x_1-x_2)$$

and A^0 and A^d the creation and annihilation parts, respectively, of the field.

To shorten the argument we present the proof of the validity of the expansion (5.1) for two equal scalar fields in three-dimensional space-time, the generalization to higher dimensions being clear.

Choose a frame in which the external-momentum configuration is such that

$$P=(0,0,P_2)\,,\quad q=(q_0,q_1,0)\,,$$
$$P_2^{\,2}=q_0^{\,2}-q_1^{\,2}=q^2\,. \quad (5.4)$$

For fixed P the matrix elements of the irreducible conformal tensors have only two independent components, which in the frame (5.4) can be chosen to be

$$\langle p_2|C^{(+n)}(0)|p_1\rangle=(q_0+q_1)^n\,,$$
$$\langle p_2|C^{(-n)}(0)|p_1\rangle=(q_0-q_1)^n\,. \quad (5.5)$$

Introducing for $x^2>0$, $x_0>0$

$$x_0=(x_0^{\,2}-x_1^{\,2})^{1/2}\cosh\theta_0\,,\quad q_0=(q^2)^{1/2}\cosh\theta\,,$$
$$x_1=(x_0^{\,2}-x_1^{\,2})^{1/2}\sinh\theta_0\,,\quad q_1=(q^2)^{1/2}\sinh\theta\,, \quad (5.6)$$

our expansion problem amounts to writing

$$\langle p_2|A^0(x_2)A^d(x_1)|p_1\rangle=\sum_{-\infty}^{\infty}a_n\,e^{n\theta}[(q^2)^{1/2}]^n \quad (5.7)$$

and using

$$\exp\{-i[(x_0^{\,2}-x_1^{\,2})^{1/2}(-P^2)^{1/2}\cosh(\theta-\theta_0)]\}$$
$$=\sum_{-\infty}^{\infty}e^{n(\theta-\theta_0)}(i)^n J_{|n|}((x_0^{\,2}-x_1^{\,2})^{1/2}(-P^2)^{1/2})\,, \quad (5.8)$$

where we have

$$a_n=e^{-n\theta_0}e^{-iPX}(i)^n\times\frac{J_{|n|}((x_0^{\,2}-x_1^{\,2})^{1/2}(-P^2)^{1/2})}{(2\pi)^2(-P^2)^n}\,. \quad (5.9)$$

Expansion (5.7) can be easily covariantized as

$$\langle p_2|A^0(x_2)A^d(x_1)|p_1\rangle=\sum_n a^{\mu_1\cdots\mu_n}(x,P)\times\langle p_2|C_{\mu_1\cdots\mu_n}(x)|p_1\rangle \quad (5.10)$$

for $x^2>0$, p_1 and p_2 arbitrary with

$$a^{\mu_1\cdots\mu_n}(x,P)=(-1)^n\frac{2^{2n+1}}{(2\pi)^2}\,\frac{J_{2n}((\hat{x}^2)^{1/2}(-P^2)^{1/2})}{(\hat{x}^2)^n(-P^2)^n}\times(\hat{x}^{\mu_1}\cdots\hat{x}^{\mu_n}-\mathrm{Tr}')\,, \quad (5.11)$$

where

$$\hat{x}^\mu=x^\mu-\frac{P^\mu(P\cdot x)}{P^2}\,, \quad (5.12)$$

so that finally for $x^2>0$, $x_0>0$

$$\langle 0|A(x_3)A^0(x_2)A^d(x_1)A(x_4)|0\rangle=\sum_N\int K^{[N]}(x_2-x',x_1-x')\langle 0|A(x_3)\,C^{[N]}(x')\,A(x_4)|0\rangle\,d^3x'\,, \quad (5.13)$$

with

$$\int_{x^2>0,\,P^2<0,\,x_0>0}K^{\mu_1\cdots\mu_n}(x_2-x',x_1-x')\,e^{-iP\cdot x'}d^3x'=e^{-iP\cdot X}\,a^{\mu_1\cdots\mu_n}(x,P)\,. \quad (5.14)$$

The derivation for x^2, $x_0<0$ is similar.

In four-dimensional space-time the analog of (5.12) will involve spherical Bessel functions. On the other hand it is readily seen for free fields that the kernel one would obtain in analogy to (3.2) by imposing term by term conformal invariance in (5.1) leads to Hankel functions instead of Bessel functions.

We conclude that although a general operator expansion of the form (5.1) might exist (and we suspect that its existence already follows from the vacuum expansion), the determination of the corresponding kernels cannot, for ξ_b and ξ_c different from zero, be obtained without a more detailed consideration of the 4-point function. Contrary to what happens in the vacuum expansion one cannot in the more general case reduce the 4-point function problem to a 3-point function by considering the limit $x_3 \to x_4$.

ACKNOWLEDGMENTS

The major part of this work was performed while one of us (J.A.S.) was visiting the Freie Universität, Berlin, under the sponsorship of a Brazilian Research Council–R. F. A. Jülich Scientific exchange program. We thank all those who made this visit possible.

APPENDIX

In this appendix we want to show that the vacuum expansion (3.1) reproduces the well-known conformal invariant three-point function in the whole Minkowski space; i.e.,

$$\langle 0|C^*(z)A^{\xi_a}(0)B(x)|0\rangle \equiv \frac{N_{abc}}{[-(x+i\epsilon)^2]^{\lambda_1}}\int d^4t\,\frac{[-(t-i\epsilon)^2]^{\lambda_3-2}[-(t+i\epsilon)^2]^{d_c-2}}{[-(t-x-i\epsilon)^2]^{\lambda_3}[-(t-z+i\epsilon)^2]^{d_c}}$$
$$= g_{abc}[-(x-z+i\epsilon)^2]^{-\delta_1}[-(x+i\epsilon)^2]^{-\delta_3}[-(z-i\epsilon)^2]^{-\delta_2}. \tag{A1}$$

As a by-product of this calculation we also obtain the normalization constant N_{abc}.

As already indicated in Sec. III the basic idea is to calculate the integral for a certain complex region in x and z, and then extend the result by analytic continuation to all Minkowski vectors x, z. From simple power counting we can easily find a complex region in λ_3 and d_c containing some part of the real axes such that the integral of (A1) is analytic in the direct product of the open backward tube (in z) and the open forward tube (in x) and, moreover, is invariant under the proper orthochronous Lorentz group L_+ on its real boundary. Hence we may restrict ourselves to calculate the integral for complex vectors x and z such that

$$\begin{aligned}&\mathrm{Im}z\in \overline{V}_-,\quad \vec{z}=0,\quad z_0\neq 0,\\ &\mathrm{Im}x\in \overline{V}_+,\quad x_1=x_2=0,\quad x_3\neq 0,\quad x_0\neq 0.\end{aligned} \tag{A2}$$

Introducing polar coordinates for $\vec{t}$ the integral reads

$$\langle 0|C^*(z)A^{\xi_a}(0)B(x)|0\rangle|_{\vec{z}=0;\,x_1=x_2=0}$$
$$=\frac{2\pi N_{abc}}{[-(x+i\epsilon)^2]^{\lambda_1}}\int_{-\infty}^{+\infty}dt_0\int_0^\infty dr\,r^2\int_0^\pi d\theta\sin\theta\,\frac{\{-[(t_0-i\epsilon)^2-r^2]\}^{\lambda_3-2}\{-[(t_0+i\epsilon)^2-r^2]\}^{d_c-2}}{\{-[(t_0-i\epsilon-x_0)^2-r^2-x_3{}^2+2rx_3\cos\theta]\}^{\lambda_3}[-(t_0-i\epsilon-z_0)^2-r^2]^{d_c}}. \tag{A3}$$

For all complex x_0, x_3 such that

$$\left|\arg\frac{(t_0-i\epsilon-x_0)^2-(r-x_3)^2}{(t_0-i\epsilon-x_0)^2-(r+x_3)^2}\right|<\pi,$$

the integral

$$I\equiv\int_0^\pi d\theta\sin\theta[r^2+x_3{}^2-(t_0-x_0-i\epsilon)^2-2rx_3\cos\theta]^{-\lambda_3}$$
$$=\frac{2}{\{-[(t_0-i\epsilon-x_0)^2-(r+x_3)^2]\}^{\lambda_3}}\int_0^1 d\omega\left[1-\frac{4rx_3}{(r+x_3)^2-(t_0-x_0-i\epsilon)^2}\right]^{-\lambda_3}$$

is just a special case of the well-known expression[16]

$$\int_0^1 dx(1-x)^\nu x^{s-1}(1+ax)^\mu = B(\nu+1,s)F(-\mu,s;\nu+s+1;-a)\,,$$

$$\text{Re}\,\nu > -1,\quad \text{Re}\,s>0,\quad |\arg(1+a)|<\pi\,. \tag{A4}$$

Applying one of Kummer's relations[17] and the equation

$$F(a,b;c;z) = (1-z)^{-b}F\left(c-a,b;c;\frac{z}{z-1}\right),$$

it follows that

$$I = [(\lambda_3-1)2rx_3]^{-1}(\{-[(t_0-i\epsilon-x_0)^2-(r-x_3)^2]\}^{1-\lambda_3} - \{-[(t_0-i\epsilon-x_0)^2-(r+x_3)^2]\}^{1-\lambda_3})$$

and therefore

$$\langle 0|C^*(z)A(0)B(x)|0\rangle|_{\vec z=0;\,x_1=x_2=0} = \frac{2\pi(1-\lambda_3)^{-1}N_{abc}}{[-(x+i\epsilon)^2]^{\lambda_1}x_3}\int_{-\infty}^{+\infty}dt_0\int_0^\infty dr\, r\,\frac{\{-[(t_0-i\epsilon)^2-r^2]\}^{\lambda_3-2}\{-[(t_0+i\epsilon)^2-r^2]\}^{d_c-2}}{\{-[(t_0-z_0+i\epsilon)^2-r^2]\}^{d_c}}$$
$$\times(\{-[(t_0-i\epsilon-x_0)^2-(r-x_3)^2]\}^{1-\lambda_3} - \{-[(t_0-i\epsilon-x_0)^2-(r+x_3)^2]\}^{1-\lambda_3})\,.$$

Next we introduce new variables

$$\begin{aligned}
&u\equiv t_0+r,\quad v\equiv t_0-r,\quad u-v=2r\geq 0\,,\\
&\xi\equiv x_0+x_3,\quad \eta\equiv x_0-x_3,\quad \xi-\eta=2x_3\,,\\
&(t_0-x_0-i\epsilon)^2-(r-x_3)^2=(u-\xi-i\epsilon)(v-\eta-i\epsilon)\,,\\
&(t_0-x_0-i\epsilon)^2-(r+x_3)^2=(u-\eta-i\epsilon)(v-\xi-i\epsilon)\,,\\
&dt_0\,dr=\tfrac{1}{2}du\,dv\,;
\end{aligned}\tag{A5}$$

$$\langle 0|C^*(z)A(0)B(x)|0\rangle|_{\vec z=0;\,x_1=x_2=0}$$
$$=\frac{\pi(\lambda_3-1)^{-1}N_{abc}}{2[-(x-i\epsilon)^2]^{\lambda_1}(\xi-\eta)}\int_{-\infty}^{+\infty}dv\int_v^{+\infty}du(u-v)\frac{[-(u-i\epsilon)(v-i\epsilon)]^{\lambda_3-2}[-(u+i\epsilon)(v+i\epsilon)]^{d_c-2}}{[-(v-z_0+i\epsilon)(u-z_0+i\epsilon)]^{d_c}}$$
$$\times\{[-(u-\xi-i\epsilon)(v-\eta-i\epsilon)]^{1-\lambda_3}-[-(u-\eta-i\epsilon)(v-\xi-i\epsilon)]^{1-\lambda_3}\}\,.$$

Now the essential point is to split the double integral into a sum of two equal terms and interchange the order of integrations in one of them, which results in

$$\int_{-\infty}^{+\infty}dv\int_v^{+\infty}du\cdots=\tfrac{1}{2}\int_{-\infty}^{+\infty}dv\int_v^{+\infty}du\cdots+\tfrac{1}{2}\int_{-\infty}^{+\infty}du\int_{-\infty}^{u}dv\cdots\,.$$

Relabeling the integration variables of the second term and using the distribution identity

$$[-(u\pm i\epsilon)(v\pm i\epsilon)]^\lambda = e^{\pm i\pi\lambda}[-(u\pm i\epsilon)]^\lambda[-(v\pm i\epsilon)]^\lambda \tag{A6}$$

finally leads to the almost-factorized expression

$$\langle 0|C^*(z)A(0)B(x)|0\rangle|_{\vec z=0;\,x_1=x_2=0} = \frac{\pi N_{abc}}{4(\lambda_3-1)[-(x+i\epsilon)^2]^{\lambda_1}(\xi-\eta)}$$
$$\times\left\{\int_{-\infty}^{+\infty}dv\frac{[-(v-i\epsilon)]^{\lambda_3-2}[-(v+i\epsilon)]^{d_c-2}}{[-(v-\eta-i\epsilon)]^{\lambda_3-1}[-(v-z_0+i\epsilon)]^{d_c}}\right.$$
$$\left.\times\int_{-\infty}^{+\infty}du(v-u)\frac{[-(u-i\epsilon)]^{\lambda_3-2}[-(u+i\epsilon)]^{d_c-2}}{[-(u-\xi-i\epsilon)]^{\lambda_3-1}[-(u-z_0+i\epsilon)]^{d_c}}-(\eta\leftrightarrow\xi)\right\}. \tag{A7}$$

Hence all that remains to be done is to calculate the simple integrals

$$I^n_{\lambda_3, d_c}(\xi, z_0) \equiv \int_{-\infty}^{+\infty} du\, u^n \frac{[-(u-i\epsilon)]^{\lambda_3-2}[-(u+i\epsilon)]^{d_c-2}}{[-(u-\xi-i\epsilon)]^{\lambda_3-1}[-(u-z_0+i\epsilon)]^{d_c}} \tag{A8}$$

for $n=0,1$.

The first important observation is the following:

Remark I. For $n=0,1$ and all (λ_3, d_c) from the set

$$\Lambda_n \equiv \{(\lambda_3, d_c) \in \mathrm{C}\times\mathrm{C}\,|\,\mathrm{Re}(\lambda_3+d_c) > 3-n,\ \ \mathrm{Re}\lambda_3 < 2\}$$

the integral $I^n_{\lambda_3, d_c}(\xi, z_0)$ is analytic in both variables (ξ, z_0) in the open set

$$\{\mathrm{C}\times\mathrm{C}\,|\,\mathrm{Im}\xi>0, \mathrm{Im}\, z_0<0\}.$$

If ξ is real, $I^n_{\lambda_3, d_c}(\xi, z_0)$ is still analytic in z_0 for $\mathrm{Im}z_0 < 0$ and all $(\lambda_3, d_c) \in \Lambda_n$.

For real ξ the integrand has two cuts just above some part of the real axis and two other ones in the lower half plane. The discontinuity of the integrand at the *two* upper cuts reads

$$\left\{\frac{[-(u-i\epsilon)]^{\lambda_3-2}}{[-(u-\xi-i\epsilon)]^{\lambda_3-1}}\right\}_{\downarrow} - \left\{\frac{[-(u-i\epsilon)]^{\lambda_3-2}}{[-(u-\xi-i\epsilon)]^{\lambda_3-1}}\right\}_{\uparrow} = \frac{|u|^{\lambda_3-2}}{|u-\xi|^{\lambda_3}} \times \begin{cases} 0 \ \text{ for } u<\min\{0,\xi\} \\ 2i\sin\lambda_3\pi \text{ for } \min\{0,\xi\} \le u \le \max\{0,\xi\} \\ 0 \ \text{ for } u>\max\{0,\xi\}. \end{cases} \tag{A9}$$

Since, moreover, for $n=0,1$ the integrand is of order $O(|u|^{-2})$ at infinity, (A8) may be rewritten as

$$I^n_{\lambda_3, d_c}(\xi, z_0) = -2i\,\sin\lambda_3\pi \int_{\min\{0,\xi\}}^{\max\{0,\xi\}} du \frac{|u|^{\lambda_3-2}u^n}{|u-\xi|^{\lambda_3-1}} \frac{[-(u+i\epsilon)]^{d_c-2}}{[-(u-z_0+i\epsilon)]^{d_c}}. \tag{A10}$$

Case I. For $\xi>0$

$$I^n_{\lambda_3, d_c}(\xi, z_0)|_{\xi>0} = -2i\sin\lambda_3\pi\, e^{-i\pi d_c}\xi^{d_c+n-2}(z_0)^{-d_c}\int_0^1 dt \frac{t^{\lambda_3+d_c+n-4}}{(1-t)^{\lambda_3-1}[1-(\xi/z_0)t]^{d_c}}. \tag{A11}$$

Remark II. Obviously the integral on the right-hand side, and therefore also $I^n_{\lambda_3, d_c}(\xi, z_0)$, possesses for all $(\lambda_3, d_c) \in \Lambda_n$ an analytic continuation into the entire z_0 plane cut along the real line $-\infty < z_0 \le \xi$.

For $|z_0| > \xi$ and $(\lambda_3, d_c) \in \Lambda_n$ the integral can be explicitly calculated[18]:

$$\begin{aligned} I^n_{\lambda_3, d_c}(\xi, z_0)|_{\xi>0} &= -2i\sin\lambda_3\pi\, e^{-i\pi d_c}B(\lambda_3+d_c+n-3, 2-\lambda_3)\xi^{d_c+n-2}(z_0)^{-d_c}F(d_c, \lambda_3+d_c+n-3; d_c+n-1; \xi/z_0) \\ &= -2i\sin\lambda_3\pi\, e^{-i\pi d_c}B(\lambda_3+d_c+n-3, 2-\lambda_3)\xi^{d_c+n-2}(z_0-\xi)^{2-\lambda_3-d_c}(z_0)^{\lambda_3-2} \\ &\quad\times F(n-1, 2-\lambda_3; d_c+n-1; \xi/z_0). \end{aligned} \tag{A12}$$

Since the hypergeometric function is in fact a polynomial of degree $n-1$, and in view of remark II both sides of (A12) are analytic in $z_0 \in \mathrm{C}\setminus\{z_0 \in \mathrm{R}\,|\,-\infty \le z_0 \le \xi\}$ for $(\lambda_3, d_c) \in \Lambda_n$ and $n=0,1$. Since both agree on the real axis $\xi < z_0 < +\infty$, Eq. (A12) holds for all complex z_0 from the cut plane

$$\mathrm{C}\setminus\{z_0 \in \mathrm{R}\,|\,-\infty \le z_0 \le \xi\}.$$

Case II. $\xi<0$. By exactly the same arguments as before it follows from (A10)

$$\begin{aligned} I^n_{\lambda_3, d_c}(\xi, z_0)|_{\xi<0} &= -2i\sin\lambda_3(-1)^n B(\lambda_3+d_c+n-3, 2-\lambda_3)(-\xi)^{d_c+n-2}(z_0-\xi)^{2-\lambda_3-d_c}(z_0)^{\lambda_3-2} \\ &\quad\times F(n-1, 2-\lambda_3; d_c+n-1; \xi/z_0) \end{aligned}$$

for $\xi<0$,

$$z_0 \in \mathrm{C}\setminus\{z_0 \in \mathrm{R}\,|\,-\infty \le z_0 \le 0\}, \quad (\lambda_3, d_c) \in \Lambda_n, \quad n=0,1.$$

Putting both cases together we find by means of the identity

$$[-(\xi \pm i\epsilon)]^{\lambda} = \exp[\lambda \ln|\xi| + i\lambda \arg(-\xi \mp i\epsilon)]$$

$$= |\xi|^{\lambda} \times \begin{cases} e^{\mp i\pi\lambda} & \text{for } \xi > 0\,, \\ 1 & \text{for } \xi < 0\,, \end{cases}$$

that

$$I^n_{\lambda_3, d_c}(\xi, z_0) = \int_{-\infty}^{+\infty} du\, u^n \frac{[-(u-i\epsilon)]^{\lambda_3-2}[-(u+i\epsilon)]^{d_c-2}}{[-(u-\xi-i\epsilon)]^{\lambda_3-1}[-(u-z_0+i\epsilon)]^{d_c}}$$
$$= 2i\pi(\lambda_3-1)(-1)^n \frac{\Gamma(\lambda_3+d_c+n-3)}{\Gamma(d_c+n-1)\Gamma(\lambda_3)} e^{-i\pi\lambda_3} \frac{[-(\xi-i\epsilon)]^{d_c+n-2}(-z_0)^{\lambda_3-2}}{[-(\xi-z_0)]^{\lambda_3+d_c-2}} F(n-1, 2-\lambda_3; d_c+n-1; \xi/z_0) \tag{A13}$$

for

$$(\xi, z_0) \in \mathrm{R} \times \{\mathrm{C} \setminus \{z_0 \in \mathrm{R} \mid -\infty \leq z_0 \leq \max(0,\xi)\}\}, \quad (\lambda_3, d_c) \in \Lambda_n, \quad n = 0, 1\,.$$

In the next step we have to extend the domain of validity of (A13) to complex ξ. According to remark I the function

$$\hat{I}^n_{\lambda_3, d_c}(\xi, z_0) \equiv [-(\xi+i\epsilon)]^{2-d_c-n} I^n_{\lambda_3, d_c}(\xi, z_0) \tag{A14}$$

is for $(\lambda_3, d_c) \in \Lambda_0 \supset \Lambda_1$ analytic in both variables ξ and z_0 for $(\xi, z_0) \in \mathrm{C}_+ \times \mathrm{C}_-$, with $\mathrm{C}_\pm$ being the open half planes

$$\mathrm{C}_\pm \equiv \{ z \in \mathrm{C} \mid \mathrm{Im} z \gtrless 0\}\,.$$

Moreover, from equation (A13) we find, that for any $z_0 \in \mathrm{C}_-$ and $(\lambda_3, d_c) \in \Lambda_0$ it is a continuous function on the real axis in ξ. Then from (A13) and the Phragmén-Lindelöf theorem[19] it follows that $I^n_{\lambda_3, d_c}(\xi, z_0)$ vanishes for $|\xi| \to +\infty$ if $0 \leq \arg\xi \leq \pi$, $z_0 \in \mathrm{C}_-$, and $(\lambda_3, d_c) \in \Lambda_0$. Hence we may use Cauchy's integral formula in the upper half plane to show that (A13) also holds for all $(\xi, z_0) \in \mathrm{C}_+ \times \mathrm{C}_-$. Combining this with the former result, it means that (A13) holds for $(\xi, z_0) \in \overline{\mathrm{C}}_+ \times \overline{\mathrm{C}}_-$, where $\overline{\mathrm{C}}_\pm$ denotes the closure of $\mathrm{C}_\pm$.

Inserting (A13) into (A7), applying the identity (A6) and going back to the original variables (A5) we find

$$\langle 0 | C^*(z) A(0) B(x) | 0 \rangle |_{\vec{z}=0;\; x_1 = x_2 = 0} = 2\pi^3 N_{abc} \frac{\Gamma(\lambda_3+d_c-2)\Gamma(\lambda_3+d_c-3)}{\Gamma(d_c)\Gamma(d_c-1)\Gamma(\lambda_3)\Gamma(1-\lambda_3)}$$
$$\times \{-[(x_0 - z_0 + i\epsilon)^2 - x_3{}^2]\}^{-\delta_1} \{-[(x_0 + i\epsilon)^2 - x_3{}^2]\}^{-\delta_3} [-(z_0 - i\epsilon)^2]^{-\delta_2} \tag{A15}$$

for

$$z_0 \in \overline{\mathrm{C}}_-, \quad (x_0 - x_3) \in \overline{\mathrm{C}}_+, \quad (x_0 + x_3) \in \overline{\mathrm{C}}_-,$$
$$\mathrm{Re}\,\delta_1 < -1, \quad \mathrm{Re}\,\delta_2 < 0\,.$$

The homogeneous function on the right-hand side of (A1) is, if we absorb the N_{abc} together with the Γ function into g_{abc}, obtained as the analytic continuation of (A15) in z and x. Finally, by analytic continuation in δ_1 and δ_2 one can also drop the restrictions in them up to certain poles on the real axis.

[1] S. Ferrara, R. Gatto, and A. Grillo, Nuovo Cimento Lett. 2, 1363 (1971); Phys. Rev. D 5, 3102 (1972); Phys. Lett. 36B, 124 (1971); Ann. Phys. (N.Y.) 76, 161 (1973).

[2] L. Bonora, Nuovo Cimento Lett. 3, 548 (1972); L. Bonora, S. Cicciariello, G. Sartori, and M. Tonin, in *Scale and Conformal Symmetry in Hadron Physics*, proceedings of the Advanced School of Physics, Frascati, Italy, 1972, edited by R. Gatto (Wiley, New York, 1973).

[3] A. A. Migdal and A. M. Polyakov (unpublished), cited in Ref. 4.

[4] G. Mack, Bern report, 1974 (unpublished).

[5] G. Mack and A. Salam, Ann. Phys. (N.Y.) 53, 174 (1969).

[6] S. Ferrara, R. Gatto, A. G. Grillo, and G. Parisi, Nucl. Phys. B49, 77 (1972); Nuovo Cimento Lett. 4, 115 (1972).

[7] J. A. Swieca and A. H. Völkel, Commun. Math. Phys. 29, 319 (1973).

[8] B. Schroer and J. A. Swieca, Phys. Rev. D 10, 480 (1974).

[9]M. Lüscher and G. Mack, Bern report, 1974 (unpublished).

[10]A. F. Grillo, Riv. Nuovo Cimento 3, 146 (1973); I. T. Todorov, in Cargèse Lectures in Physics, 1973 (unpublished).

[11]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1.

[12]J. Kupsch, W. Rühl, and B. C. Yunn, Ann. Phys. (N.Y.) 89, 115 (1975).

[13]B. Klaiber, *1967 Boulder Lectures in Physics, Vol. 10A: Quantum Theory and Statistical Theory*, edited by W. E. Brittin *et al*. (Gordon and Breach, New York, 1968), p. 141.

[14]J. H. Lowenstein, Commun. Math. Phys. 16, 265 (1970).

[15]G. Parisi, Rome report, 1972 (unpublished); C. G Callan, Jr., and D. J. Gross, Phys. Rev. D 8, 4383 (1973).

[16]*Tables of Integral Transforms* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1954), Vol. 1, p. 310.

[17]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1, p. 107, Eq. (34).

[18]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965), p. 286.

[19]E. C. Titchmarsh, *The Theory of Functions* (Oxford Univ. Press, London, 1960), p. 176.